Wreath Products in the Unit Group of Modular Group Algebras of 2-groups of Maximal Class

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Abstract

We study the unit group of the modular group algebra KG, where G is a 2-group of maximal class.

We prove that the unit group of KG possesses a section isomorphic to the wreath product of a group of order two with the commutator subgroup of the group G.

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1 Introduction

Let p be a prime number, G be a finite p-group and K be a field of characteristic p. Denote by $\Delta = \Delta_K(G)$ the augmentation ideal of the modular group algebra KG. The group of normalized units $U(G) = U(KG)$ consists of all elements of the type $1 + x$, where $x \in \Delta$. Our further notation follows [20].

Define Lie-powers $KG^{[n]}$ and $KG^{(n)}$ in KG: $KG^{[n]}$ is two-sided ideal, generated by all (left-normed) Lie-products $[x_1, x_2, \cdots, x_n], x_i \in KG$, and $KG^{(n)}$ is defined inductively: $KG^{(1)} = KG, KG^{(n+1)}$ is the associative ideal generated by $[KG^{(n)}, KG]$. Clearly, for every $n \quad KG^{(n)} \supseteq KG^{[n]}$, but equality need not hold.

For modular group algebras of finite p-groups $KG^{(|G'|+1)} = 0 \quad$ [24]. Then in our case finite lower and upper Lie nilpotency indices are defined:

$$t_L(G) = min\{n : KG^{[n]} = 0\}, \qquad t^L(G) = min\{n : KG^{(n)} = 0\}.$$

It is known that $t_L(G) = t^L(G)$ for group algebras over the field of characteristic zero [19], and for the case of characteristic $p > 3$ their coincidence was proved by A. Bhandari and I. B. S. Passi [3].

Consider the following normal series in $U(G)$:

$$U(G) = 1 + \Delta \supseteq 1 + \Delta(G') \supseteq 1 + \Delta^2(G') \supseteq 1 + \Delta^{t(G')}(G'),$$

where $t(G')$ is the nilpotency index of the augmentation ideal of KG'.

An obvious question is whether does exist a refinement for this normal series. There were two conjectures relevant to the question above.

The first one, as it was stated in [20], is attributed to A. A. Bovdi and consists in the equality $\operatorname{cl} U(G) = t(G')$, i.e. this normal series doesn't have a refinement. In particular, C. Baginski [1] proved that $\operatorname{cl} U(G) = p$ if $|G'| = p$ (in case of cyclic commutator subgroup $t(G') = |G'|$). A. Mann and A. Shalev proved that $\operatorname{cl} U(G) \leq t(G')$ for groups of class two [16].

The second conjecture was suggested by S. A. Jennings [13] in a more general context, and in our case it means that $\operatorname{cl} U(G) = t_L(G) - 1$. Here N. Gupta and F. Levin proved inequality $\leq$ in [11].

The first conjecture was more attractive and challenging, since methods for the systematic computation of the nilpotency index of the augmentation ideal $t(G')$ were more known than such ones for the calculation of the lower Lie nilpotency index (for key facts see, for example, [12], [15], [18], [21], [24]).

Moreover, A. Shalev [20] proved that these two conjectures are incompatible in general case, although $t(G') = t_L(G) - 1$ for some particular families of groups, including 2-groups of maximal class. Later using computer Coleman managed to find counterexample to Bovdi's conjecture (cf. [25]), and the final effort in this direction was made bu X. Du [10] in his proof of Jennings conjecture.

Study of the structure of the unit group of group algebra and its nilpotency class raised a number of questions of independent interest, in particular, about involving of different types of wreath products in the unit group (as a subgroup or as a section).

In [9] D. Coleman and D. Passman proved that for non-abelian finite p-group G a wreath product of two groups of order p is involved into $U(KG)$. Later this result was generalized by A. Bovdi in [4]. Among other related results it is worth to mention [16], [17], [23].

It is also an interesting question whether $U(KG)$ posesses a given wreath product as a subgroup or only as a section, i.e. as a factor-group of a certain subgroup of $U(KG)$. Baginski in [1] described all p-groups, for which $U(KG)$ does not contain a subgroup isomorphic to the wreath product of two groups of order p for the case of odd p, and the case of $p = 2$ was investigated in [7].

The question whether $U(G)$ possesses a section isomorphic to the wreath product of a cyclic group C_p of order p and the commutator subgroup of G was stated by A. Shalev in [20]. Since the nilpotency class of the wreath product $C_p \wr H$ is equal to $t(H)$ - the nilpotency index of the augmentation ideal of KH [8], this question was very useful for the investigation of the first conjecture. In [22] positive answer was given by A. Shalev for the case of odd p and a cyclic commutator subgroup of G.

The present paper is aimed to extend the last result on 2-groups of maximal class, proving that if G is such a group then the unit group of KG possesses a section isomorphic to the wreath product of a group of order two with the commutator subgroup of the group G. We prove the following main result.

Theorem 1 *Let K be a field of characteristic two, G be a 2-group of maximal class. Then the wreath product $C_2 \wr G'$ of a cyclic group of order two and the commutator subgroup of G is involved in $U(KG)$.*

2 Preliminaries

We consider 2-groups of maximal class, namely, the dihedral, semidihedral and generalized quaternion groups, which we denote by D_n, S_n and Q_n respectively. They are given by following representations [2]:

$$\begin{aligned} D_n &= \langle a, b \,|\, a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\ S_n &= \langle a, b \,|\, a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle, \\ Q_n &= \langle a, b \,|\, a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle, \end{aligned}$$

where $n \geq 3$ (We shall consider D_3 and S_3 as identical groups).

We may assume that K is a field of two elements, since in the case of an arbitrary field of characteristic two we may consider its simple subfield and corresponding subalgebra in KG, where G is one of the groups D_n, S_n or Q_n.

Denote for $x = \sum_{g \in G} \alpha_g \cdot g$, $\alpha_g \in K$ by $\operatorname{Supp} x$ the set $\{g \in G \,|\, \alpha_g \neq 0\}$. Since K is a field of two elements, $x \in 1 + \Delta \Leftrightarrow |\operatorname{Supp} x| = 2k + 1, k \in \mathrm{N}$.

Next, for every element g in G there exists unique representation in the form $a^i b^j$, where $0 \leq i < 2^{n-1}, 0 \leq j < 2$. Then for every $x \in KG$ there exists unique representation in the form $x = x_1 + x_2 b$, where $x_i = a^{n_1} + \ldots + a^{n_{k_i}}$. We shall call x_1, x_2 *components* of x. Clearly, $x_1 + x_2 b = y_1 + y_2 b \Leftrightarrow x_i = y_i, i = 1, 2$.

The mapping $x \mapsto \bar{x} = b^{-1} x b$, which we shall call *conjugation*, is an automorphism of order 2 of the group algebra KG. An element z such that $z = \bar{z}$ will be called *self-conjugated*.

Using this notions, it is easy to obtain the rule of multiplication of elements from KG, which is formulated in the next lemma.

Lemma 1 *Let $f_1 + f_2 b, h_1 + h_2 b \in KG$. Then*

$$(f_1 + f_2 b)(h_1 + h_2 b) = (f_1 h_1 + f_2 \bar{h}_2 \alpha) + (f_2 \bar{h}_1 + f_1 h_2) b,$$

where $\alpha = 1$ for D_n and S_n, $\alpha = b^2$ for Q_n.

We proceed with a pair of technical results.

Lemma 2 *An element $z \in KG$ commute with $b \in G$ if and only if z is self-conjugated.*

Lemma 3 *If x and y are self-conjugated, then $xy = yx$.*

In the next lemma we find the inverse element for an element from $U(KG)$.

Lemma 4 *Let $f = f_1 + f_2 b \in U(KG)$. Then $f^{-1} = (\bar{f}_1 + f_2 b)R^{-1}$, where $R = f_1\bar{f}_1 + f_2\bar{f}_2\alpha$, and $\alpha = 1$ for D_n, S_n, $\alpha = b^2$ for Q_n.*

Proof. Clearly, R is a self-conjugated element of $K\langle a\rangle$ of augmentation 1, hence R is a central unit in KG. Then the lemma follows since

$$(f_1 + f_2 b)(\bar{f}_1 + f_2 b) = (\bar{f}_1 + f_2 b)(f_1 + f_2 b) = R.$$

Now we formulate another technical lemma, which is easy to prove by straightforward calculations using previous lemma.

Lemma 5 *Let $f, h \in U(G), f = f_1 + f_2 b, h = h_1 + h_2 b$, and $h = \bar{h}$ is self-conjugated. Let R and α be as in the lemma 4. Then $f^{-1}hf = t_1 + t_2 b$, where $t_1 = h_1 + h_2(f_1 f_2 + \bar{f}_1\bar{f}_2)\alpha R^{-1}, t_2 = h_2(\bar{f}_1^2 + f_2^2\alpha)R^{-1}$.*

Let us consider the mapping $\varphi(x_1 + x_2 b) = x_1\bar{x}_1 + x_2\bar{x}_2\alpha$, where α was defined in the Lemma 5. It is easy to verify that such mapping is homomorphism from $U(KG)$ to $U(K\langle a\rangle)$ and, clearly, for every x its image $\varphi(x)$ is self-conjugated. Such mapping $\varphi : U(KG) \to U(K\langle a\rangle)$ we will call *norm*. We will also say that the norm of an element x is equal to $\varphi(x)$.

3 Dihedral and Semidihedral Group

Now let G be the dihedral or semidihedral group. Note that in Lemma 5 the first component of $f^{-1}hf$ is always self-conjugated. In general, the second one need not have the same property, but it is self-conjugated in the case when $f_1 + f_2 = 1$, where $f = f_1 + f_2 b$. It is easy to check that the set

$$H(KG) = \{h_1 + h_2 b \in U(KG) \mid h_1 + h_2 = 1\}$$

is a subgroup of $U(KG)$.

Now we define the mapping $\psi : H(KG) \to U(K\langle a\rangle)$ as a restriction of $\varphi(x)$ on $H(KG)$. For convenience we also call it *norm*.

Lemma 6 $\ker\psi = C_H(b) = \{h \in H(KG) \mid hb = bh\}$.

The proof follows from the Lemma 2 and the equality $\psi(h) = 1 + h_1 + \bar{h}_1$ for $h \in H(KG)$.

Lemma 7 *$C_H(b)$ is elementary abelian group.*

Proof. $C_H(b)$ is abelian by the lemma 3. Now, $h^2 = f_1^2 + f_2^2 + (f_1 f_2 + f_1 f_2)b = f_1^2 + f_2^2 = 1 + f_1^2 + f_1^2 = 1$, and we are done.

Note that for elements $f, h \in C_H(b)$ we may obtain more simple rule of their multiplication: $(f_1 + f_2 b)(h_1 + h_2 b) = (1 + f_1 + h_1) + (f_1 + h_1)b$.

Now we consider a subgroup in G generated by $b \in G$ and $A = a + (1+a)b$. Note that $A \in H(KG)$. First we calculate the norm of $A : \psi(A) = 1 + a + \bar{a}$. Since $a^{2^{n-1}} = 1$, we have $\psi(A)^{2^{n-2}} = 1$. Then $A^{2^{n-2}}$ commute with b, and $A^{2^{n-1}} = 1$ by the Lemma 7. Smaller powers of A have non-trivial norm, so they do not commute with b. Clearly, order of A is equal to 2^{n-2} or 2^{n-1}. In the following lemma we will show that actually only the second case is possible.

Lemma 8 *Let $A = a + (1+a)b$. Then the order of A is equal to 2^{n-1}.*

Proof. We will show that the case of 2^{n-2} is impossible since $\operatorname{Supp} A^{2^{n-2}} \neq 1$. We will use formula (8) from [6], which describes 2^k-th powers of an element $x \in U(G)$:

$$x^{2^k} = x_1^{2^k} + (x_2 \bar{x}_2)^{2^{k-1}} b^{2^k} + \sum_{i=1}^{k-1} (x_2 \bar{x}_2)^{2^{i-1}} (x_1 + \bar{x}_1)^{2^k - 2^i} b^{2^i} + x_2 (x_1 + \bar{x}_1)^{2^k - 1} b.$$

Let us show that the second component of $A^{2^{n-2}}$ is non-trivial. Let us denote it by $t_2(A^{2^{n-2}}) = t_2$. By the cited above formula, $t_2 = (1+a)(a+\bar{a})^{2^{n-2}-1}$, where $\bar{a} = a^{-1}$ for the dihedral group, and $\bar{a} = a^{-1+2^{n-2}}$ for the semidihedral group.

Now we consider the case of the dihedral group. We have

$$t_2 = (1+a)(a+a^{-1})^{2^{n-2}-1} = (1+a)(a^{-1}(1+a^2))^{2^{n-2}-1} =$$

$$(1+a)a^{2^{n-2}+1}(1+a^2)^{2^{n-2}-1} = (a^{2^{n-2}+1} + a^{2^{n-2}+2})(1+a^2)^{2^{n-2}-1}.$$

Note that if $X = \langle x \rangle, x^{2^m} = 1$, then $(1+x)^{2^m - 1} = \sum_{y \in X} y = \overline{X}$, where for a set X we denote by $\overline{X}$ the sum of all its elements [2]. Thus, we have

$$(1+a^2)^{2^{n-2}-1} = 1 + a^2 + a^4 + \ldots + a^{2^{n-1}-2} = \overline{\langle a^2 \rangle} = \overline{G'}.$$

Then

$$t_2 = (a^{2^{n-2}+1} + a^{2^{n-2}+2})\overline{\langle a^2 \rangle} = a\overline{\langle a^2 \rangle} + \overline{\langle a^2 \rangle} = \overline{\langle a \rangle},$$

and for the case of the dihedral group the lemma is proved.

Now we will consider the semidihedral group. We have

$$t_2 = (1+a)(a+a^{2^{n-2}-1})^{2^{n-2}-1} = (1+a)(a(1+a^{2^{n-2}-2}))^{2^{n-2}-1} =$$

$$(a^{2^{n-2}-1} + a^{2^{n-2}})(1+a^{2^{n-2}-2})^{2^{n-2}-1},$$

and the rest part of the proof is similar. Note that from $t_2(A^{2^{n-2}}) = \overline{\langle a \rangle}$ we can immediately conclude that its first component is $1 + \overline{\langle a \rangle}$, since $A \in H(G)$.

To construct a section isomorphic to the desired wreath product, first we take elements $b, b^A, b^{A^2}, \cdots, b^{A^{2^{n-2}-1}}$. For every k we have $(b^{A^k})^2 = 1$. By the Lemma 5 all elements b^{A^k} are self-conjugated, since $A \in H(KG)$, and they commute each with other by the Lemma 3. So, we get the next lemma.

Lemma 9 *$\langle b, b^A, b^{A^2}, \cdots, b^{A^{2^{n-2}-1}} \rangle$ is elementary abelian subgroup.*

Now we can obtain elements b^{A^k}, using the Lemma 5.

Lemma 10 *Let $A = a + (1+a)b, \quad R = \psi(A) = 1 + a + \bar{a}$. Then*

$$b^{A^k} = 1 + R^k + R^k b, \qquad k = 1, 2, ..., 2^{n-2}.$$

Proof. First we obtain b^A by Lemma 5 with $h_1 = 0, h_2 = 1, f_1 = a, f_2 = 1+a$. We get

$$b^A = (a(1+a) + \bar{a}(1+\bar{a}))R^{-1} + (\bar{a}^2 + (1+a)^2)R^{-1}b =$$

$$= (a+\bar{a}+a^2+\bar{a}^2)R^{-1} + (1+a^2+\bar{a}^2)R^{-1}b = (R+R^2)R^{-1} + R^2R^{-1}b = 1+R+Rb.$$

Now let $b^{A^k} = 1 + R^k + R^k b$. Using the same method for $h_1 = 1 + R^k, h_2 = R^k$, we get $b^{A^{k+1}} = 1 + R^{k+1} + R^{k+1}b$, as required.

Lemma 11 *There exists following direct decomposition:*

$$\langle b, b^A, b^{A^2}, \cdots, b^{A^{2^{n-2}-1}} \rangle = \langle b \rangle \times \langle b^A \rangle \times \langle b^{A^2} \rangle \times \cdots \times \langle b^{A^{2^{n-2}-1}} \rangle$$

Proof. We need to verify that the product of the form $b^{i_0}(b^A)^{i_1} \cdots (b^{A^k})^{i_k}$, where $k = 2^{n-2} - 1$, $i_m \in \{0, 1\}$ and not all i_m are equal to zero, is not equal to $1 \in G$. Clearly, multiplication by b only permute components. So, we may consider only products without b and proof that they are not equal to 1 or b.

Note that b^{A^k} are self-conjugated and lies in $H(KG)$. From this follows the rule of their multiplication:

$$(1 + R^k + R^k b)(1 + R^m + R^m b) = 1 + R^k + R^m + (R^k + R^m)b.$$

The product of more than two elements is calculated by the same way:

$$(b^A)^{i_1}(b^{A^2})^{i_2} \cdots (b^{A^k})^{i_k} = 1 + i_1 R + i_2 R^2 + \cdots + i_k R^k + (i_1 R + i_2 R^2 + \cdots + i_k R^k)b.$$

Put $\gamma = i_1 R + i_2 R^2 + \cdots + i_k R^k$ and $R = 1 + r$, where $r = a + \bar{a}$. Then γ could be written in the form $\gamma = \mu + r^{j_1} + \cdots + r^{j_k}$, where $\mu \in \{0, 1\}$ and $j_1 < j_2 < \cdots < j_k = i_k$. Since $(a + \bar{a})^{2^{n-2}} = 0$, r is nilpotent and its smaller powers are linearly independent, so $r^{j_1} + \cdots + r^{j_k} \neq 0$. From the other side, it is easy to see that the support of r^{j_s} does not contain 1, so $r^{j_1} + \cdots + r^{j_k} \neq 1$. Hence $\gamma \notin \{0, 1\}$, and the support of the product $(b^A)^{i_1}(b^{A^2})^{i_2} \cdots (b^{A^k})^{i_k}$ contains elements different from 1 and b, which proves the lemma.

Now we are ready to finish the proof of Theorem 1 for the dihedral and semidihedral groups. It was shown that $U(KG)$ contains the semi-direct product F of $\langle b \rangle \times \langle b^A \rangle \times \langle b^{A^2} \rangle \times \cdots \times \langle b^{A^{2^{n-2}-1}} \rangle$ and $\langle A \rangle$. As was proved above, the order of A is 2^{n-1} and its 2^{n-2}-th power commutes with b. From this follows that the factorgroup $F/\langle A^{2^{n-2}} \rangle$ is isomorphic to $C_2 \wr G'$, as required.

4 Generalized Quaternion Group

Now let G be the generalized quaternion group. First we need to calculate $\operatorname{cl} U(G)$. In fact, we need to know only $t_L(G)$, since $\operatorname{cl} U(G) = t_L(G) - 1$ [10]. Note that Theorem 2 is already known (see Theorem 4.3 in [5]), but we provide an independent proof for the generalized quaternion group.

Theorem 2 *Let G be the generalized quaternion group. Then $clU(G) = |G'|$.*

Proof. First, $\operatorname{cl} U(G) \leq |G'|$ by [26]. Now we prove that $t_L(G) \geq |G'| + 1$. To do this, we will construct non-trivial Lie-product of the length $2^{n-2} = |G'|$.

Consider Lie-product $[b, \underbrace{a, \cdots, a}_{k}]$ which we denote by $[b, k \cdot A]$. Clearly, $[b, a] = (a + a^{-1})b$, and $(a + a^{-1})$ is central in KG. It is easy to prove by induction that $[b, k \cdot a] = (a + a^{-1})^k b$, therefore the commutator

$$[b, (2^{n-2} - 1) \cdot a] = a^{2^{n-2}+1}(1 + a^2)^{2^{n-2}-1} b = a^{2^{n-2}+1}\overline{\langle a^2 \rangle} b = a\overline{\langle a^2 \rangle} b$$

does not vanish.

From the Theorem 2 it follows that $t_L(G) = t^L(G)$ since

$$\operatorname{cl} U(G) = t_L(G) - 1 \leq t^L(G) - 1 \leq |G'|,$$

confirming conjecture about equality of the lower and upper Lie nilpotency indices (cf. [3]). From this we conclude that G and $U(KG)$ have the same exponent, using the theorem from [24] about coincidence of their exponents in the case when $t^L(G) \leq 1 + (p-1)p^{e-1}$, where $p^e = \exp G$ and p is the characteristic of the field K. Note that these two statements regarding Lie nilpotency indices and exponent are also true for all 2-groups of maximal class. Using the technique described here we also may show that modular group algebras of 2-groups of maximal class are Lie centrally metabelian.

For a unit A of KG we denote by $(b, k \cdot A)$ the commutator $(b, \underbrace{A, \cdots, A}_{k})$. Now we need a pair of technical lemmas.

Lemma 12 *Let $A \in U(KG), A^{2^{n-1}} = 1, b \in G, b^{A^i} b^{A^j} = b^{A^j} b^{A^i}$ for every i, j, where $b^{A^i} = A^{-i} b A^i$. Then for every $k \in \mathbf{N}$ $(b, k \cdot A)^2 = 1$.*

Proof. We use induction by k. By straightforward calculation, $(b, A)^2 = 1$. Now, let $(b, k \cdot A) = X, X^2 = 1$. Then $(X, A) = XX^A$. Since elements $b^{A^i}, i \in \mathbf{N}$ commute each with other, X and X^A also commute, and $(XX^A)^2 = 1$.

Lemma 13 *Let $A \in U(KG), A^{2^{n-1}} = 1, b \in G, b^{A^i} b^{A^j} = b^{A^j} b^{A^i}$ for every i, j, where $b^{A^i} = A^{-i} b A^i$. Then for every $k, m \in \mathbf{N}$*

$$(b, \underbrace{A, \cdots, A}_{k}, A^{2^m}) = (b, \underbrace{A, \cdots, A}_{k+2^m}).$$

Proof. We use induction by m. First, $(b, k\cdot A, A) = (b, (k+1)\cdot A)$. Let the statement holds for some m. Consider the commutator

$$(b, k\cdot A, A^{2^{m+1}}) = (b, k\cdot A, A^{2^m})^2 (b, k\cdot A, A^{2^m}, A^{2^m}),$$

since $(x, yz) = (x, y)(x, z)(x, y, z)$. By the Lemma 12 the square of the first commutator is 1, while the second is equal to $(b, (k+2^{m+1})\cdot A)$.

This gives possibility to proof the next property of $U(KG)$.

Lemma 14 *Let $A \in U(KG), A^{2^{n-1}} = 1, b \in G, b^{A^i} b^{A^j} = b^{A^j} b^{A^i}$ for every i, j, where $b^{A^i} = A^{-i} b A^i$. Then $A^{2^{n-2}}$ commute with b.*

Proof. We will show using induction by m that the group commutator $(b, A^{2^m}) = (b, \underbrace{A, \cdots, A}_{2^m})$, so $(b, A^{2^{n-2}}) = (b, \underbrace{A, \cdots, A}_{2^{n-2}}) = 1$, since $\operatorname{cl} U(G) = 2^{n-2}$.

First, $(b, A^2) = (b, A)^2 (b, A, A) = (b, A, A)$ by the Lemma 12. Let $(b, A^{2^m}) = (b, 2^m \cdot A)$. Then $(b, A^{2^{m+1}}) = (b, A^{2^m})^2 (b, A^{2^m}, A^{2^m}) = (b, A^{2^m}, A^{2^m})$ by the Lemma 12. Using induction hypothesis and Lemma 13, we get

$$(b, A^{2^m}, A^{2^m}) = (b, 2^m \cdot A, A^{2^m}) = (b, 2^{m+1}\cdot A).$$

Let us take an element $A = a^{2^{n-3}+1} + (1+a)b$, where $a^{2^{n-1}} = 1$. Calculating $A^{-1}hA$ for self-conjugated h by the Lemma 5, we get a self-conjugated element again. The norm of A is $\varphi(A) = 1 + a^{2^{n-2}+1} + a^{2^{n-2}-1}$, so order of $\varphi(A)$ is 2^{n-2}, and from this we conclude that the order of A is great or equal to 2^{n-2}. From the other side, it is not greater then 2^{n-1}, since G and $U(KG)$ have the same exponent. Moreover, if $A^{2^{n-2}} \neq 1$, then $A^{2^{n-2}}$ commute with b by lemma 14, and it is necessary to know whether its lower powers commute with b. As in the previous section, in the following lemma we will exactly calculate the order of A.

Lemma 15 *Let $A = a^{2^{n-3}+1} + (1+a)b$. Then the order of A is equal to 2^{n-1}.*

Proof. The proof is similar to the proof of the lemma 8. We will show that $\operatorname{Supp} A^{2^{n-2}} \neq 1$, calculating the second component $t_2(A^{2^{n-2}}) = t_2$. Using the same formula from [6], we have:

$$t_2 = (1+a)(a^{2^{n-3}+1} + a^{-2^{n-3}-1})^{2^{n-2}-1} = (1+a)(a^{-2^{n-3}-1}(1+a^{2^{n-2}+2}))^{2^{n-2}-1} =$$

$$(1+a)(a^{-2^{n-3}-1})^{2^{n-2}-1}(1+a^{2^{n-2}+2})^{2^{n-2}-1} = (1+a)a^{-2^{n-3}+1}(1+a^{2^{n-2}+2})^{2^{n-2}-1} =$$

$$(a^{-2^{n-3}+1} + a^{-2^{n-3}+2})(1+a^{2^{n-2}+2})^{2^{n-2}-1}.$$

Then, $(1+a^{2^{n-2}+2})^{2^{n-2}-1} = \overline{\langle a^2 \rangle} = \overline{G'}$. From this

$$t_2 = (a^{-2^{n-3}+1} + a^{-2^{n-3}+2})\overline{\langle a^2 \rangle} = a\overline{\langle a^2 \rangle} + \overline{\langle a^2 \rangle} = \overline{\langle a \rangle},$$

and the lemma is proved.

Now we calculate elements $b^{A^k}, k = 1, 2, \cdots, 2^{n-2}$, using Lemma 5.

Lemma 16 *Let* $A = a^{2^{n-3}+1} + (1+a)b, \quad R = \varphi(A) = 1 + a^{2^{n-2}+1} + a^{2^{n-2}-1}$. *Then*

$$b^{A^k} = \beta \sum_{i=-1}^{k-1} (b^2R)^i + (b^2R)^k b, \quad k = 1, 2, ..., 2^{n-2},$$

where $\beta = a^{2^{n-3}+1} + a^{-2^{n-3}-1} + a^{2^{n-3}+2} + a^{-2^{n-3}-2}$.

Proof. Remember that for Q_n in Lemma 5 $f^{-1}hf = t_1 + t_2 b$, where

$$t_1 = h_1 + h_2(f_1 f_2 + \bar{f}_1 \bar{f}_2) b^2 R^{-1}, \qquad t_2 = h_2(\bar{f}_1^2 + f_2^2 b^2) R^{-1}.$$

First we obtain the second component. For $A = f_1 + f_2 b$ we have $\bar{f}_1^2 + f_2^2 b^2 = (a^{-2^{n-3}-1})^2 + (1+a)^2 a^{2^{n-2}} = b^2(1+a^2+a^{-2}) = b^2R^2$. Then the second component of b^A is $b^2R^2R^{-1} = b^2R$. Now it is easy to prove by induction that the second component of b^{A^k} is $(b^2)^k R^k$. From this immediately follows that $A^k, k < 2^{n-2}$, doesn't commute with b, since $\operatorname{ord} R = 2^{n-2}$.

Now we will calculate the first component. First, for the element A expression of the form $f_1 f_2 + \bar{f}_1 \bar{f}_2$ is equal to $a^{2^{n-3}+1}(1+a) + a^{-2^{n-3}-1}(1+a^{-1}) = a^{2^{n-3}+1} + a^{-2^{n-3}-1} + a^{2^{n-3}+2} + a^{-2^{n-3}-2}$, which we will denote by β. Using the formula at the beginning of the proof for $h_1 = 0, h_2 = 1$ we conclude that the first component of b^A is equal to $\beta b^2 R^{-1}$. Now let the first component of b^{A^k}, where $k < 2^{n-2} - 1$, is equal to $\beta \sum\limits_{i=-1}^{k-1} (b^2R)^i$. Taking into consideration its previously calculated second component, we obtain that the first component of $b^{A^{k+1}}$ is equal to

$$\beta \sum_{i=-1}^{k-1} (b^2R)^i + \beta (b^2R)^k b^2 R^{-1} = \beta \sum_{i=-1}^{k} (b^2R)^i.$$

Now we are ready to construct the subgroup, whose factorgroup is isomorphic to the desired wreath product. Let us consider the subgroup F_1 in $U(G)$:

$$F_1 = \langle b, b^A, b^{A^2}, \cdots, b^{A^{2^{n-2}-1}} \rangle \langle A \rangle,$$

where $b \in G, A = a^{2^{n-3}+1} + (1+a)b$, $A^{2^{n-2}}$ is the minimal power of A which commutes with b. Further, the subgroup $\langle b, b^A, b^{A^2}, \cdots, b^{A^{2^{n-2}-1}} \rangle$ is abelian, and the intersection of subgroups $\langle b \rangle, \langle b^A \rangle, \cdots, \langle b^{A^{2^{n-2}-1}} \rangle$ is $\langle b^2 \rangle$. Moreover, the order of A is 2^{n-1}.

Let us take F_2 as a factorgroup of the group F_1 as follows:

$$F_2 = F_1 / \langle b^2 \rangle \langle A^{2^{n-2}} \rangle.$$

It is clear, that $\operatorname{cl} F_2 \leq 2^{n-2} = \operatorname{cl} U(G)$. If we will show that actually we have equality $\operatorname{cl} F_2 = 2^{n-2} = \operatorname{cl}(C_2 \wr G')$, then from this it will follow that $F_2 \cong C_2 \wr G'$.

Let $M = C_2 \wr G'$ and $\operatorname{cl} F_2 = 2^{n-2} = \operatorname{cl} M$. Let us assume that $F_2 \ncong M$. Then there exists such normal subgroup $N \lhd M$, that $M/N \cong F_2$, since there

exists a homomorphism $M \to F_2$, which is induced by mapping of generators of M into F_2. Since $|Z(M)| = 2$, $N \triangleleft M$, then $N \cap Z(M) \neq \emptyset$, so $Z(M) \subseteq N$. In this case the nilpotency class $\operatorname{cl} F_2$ should be less then $\operatorname{cl} M$, and we will get a contradiction.

To obtain the lower bound for the nilpotency class $\operatorname{cl} F_2$ we will show that the commutator $(b, \underbrace{A \dots A}_{2^{n-2}-1})$ in $U(G)$ does not belong to the subgroup $\langle b^2 \rangle \langle A^{2^{n-2}} \rangle$, so its image in F_2 is nontrivial. By the lemma 13 $(b, \underbrace{A \dots A}_{2^{n-2}-1}) = (b, \underbrace{A \dots A}_{2^{n-3}-1}, A^{2^{n-3}}) = (b, \underbrace{A \dots A}_{2^{n-4}-1}, A^{2^{n-4}}, A^{2^{n-3}}) = \cdots = (b, A, A^2, A^4, \cdots, A^{2^{n-4}}, A^{2^{n-3}})$, and we obtain more simple commutator of the length $n-1$. Further, $(b, A) = b^{-1}b^A = bb^A$, then $(b, A, A^2) = bb^A b^{A^2} b^{A^3}$, and, by induction,

$$(b, A, A^2, \cdots, A^{2^{n-3}}) = bb^A b^{A^2} \cdots b^{A^{2^{n-2}-1}} = (bA^{-1})^{2^{n-2}} A^{2^{n-2}}.$$

It remains to show that $(bA^{-1})^{2^{n-2}}$ does not contained in $\langle b^2 \rangle \langle A^{2^{n-2}} \rangle$. Note that $(bA^{-1})^{-1} = Ab^3$, $(Ab^3)^{2^{n-2}} = (Ab)^{2^{n-2}}$.

By the Lemma 15 the second component of $A^{2^{n-2}}$ is equal to $\overline{\langle a \rangle}$. Note that it is not changed under multiplication of $A^{2^{n-2}}$ by b^2. The same method could be used for calculation of $(Ab)^{2^{n-2}}$. We have

$$Ab = (1+a)b^2 + a^{2^{n-3}+1}b = (a^{2^{n-2}} + a^{2^{n-2}+1}) + a^{2^{n-3}+1}b.$$

Then by the formula from [6] the second component of $(Ab)^{2^{n-2}}$ is equal to

$$a^{2^{n-3}+1}(a^{2^{n-2}+1} + a^{2^{n-2}-1})^{2^{n-2}-1} = a^{2^{n-3}+1}(a^{2^{n-2}-1}(1+a^2))^{2^{n-2}-1} =$$

$$a^{2^{n-3}+1}(a^{2^{n-2}-1})^{2^{n-2}-1}(1+a^2)^{2^{n-2}-1} = a^{2^{n-3}+2}\overline{\langle a^2 \rangle} = \overline{\langle a^2 \rangle}.$$

Thus, support of the second component of $(Ab)^{2^{n-2}}$ does not coincide with the support of the second component of $(A)^{2^{n-2}}$ and does not changes under multiplication of $(Ab)^{2^{n-2}}$ by b^2. From this we conclude that $(Ab)^{2^{n-2}} \notin \langle b^2 \rangle \langle A^{2^{n-2}} \rangle$. This proves that the commutator $(b, \underbrace{A \dots A}_{2^{n-2}-1})$ also does not lies there. That is why $\operatorname{cl} F_2 = 2^{n-2} = \operatorname{cl}(C_2 \wr G')$, and $F_2 \cong C_2 \wr G'$, so the theorem is proved.

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